CONCERNING THE DEFINITION BY A SYSTEM OF FUNCTIONAL PROPERTIES OF THE FUNCTION $f(z) = \dfrac{\sin \pi z}{\pi}$.

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Introduction.

The definition of a function by a system (or, preferably, by various equivalent systems) of characteristic functional properties is of fundamental importance in the theory of that function, whether that theory be considered in itself or with respect to its application in the investigation of other functions or classes of functions.

I have not found in the literature consulted any such functional definition of the function $f(z) = \dfrac{\sin \pi z}{\pi}$, except that connected with its differential equation.

In the first part of this paper a functional definition is established for the function in question. In the second part this definition is used (for which purpose, indeed, this particular definition was hit upon) to effect a determination of the external exponential factor in the expression of the function as a Weierstrassian infinite product of primary factors.

Part I.

Definition of the function $f(z) = \dfrac{\sin \pi z}{\pi}$ *by a system of characteristic functional properties.*

Theorem.* *There exists one and only one function* $f(z)$, $f(z) = \dfrac{\sin \pi z}{\pi}$, *which possesses the following functional properties :*

($\mathbf{A}_1$) $f(z)$ *is a (transcendental) integral function of the complex variable* z.

($\mathbf{A}_2$) $f(z)$ *has as its complete system of zeros* $z = m = 0, \pm 1, \pm 2, \ldots$, *the multiplicity of each zero being unity.*

$$(\mathbf{A}_3) \qquad \lim_{z=0} \frac{f(z)}{z} = +1\,, \quad \text{or} \quad \left[\frac{d}{dz} f(z)\right]_{z=0} = +1\,.$$

* This theorem was embodied in a paper, bearing the same title as this paper, read August 15, 1894, in Brooklyn before the American Mathematical Society at its first summer meeting. In that proof (**C**) and two other properties (**D**, **E**), direct consequences of (**A**, **B**, **C**),

$$(\mathbf{D}) \qquad f(\tfrac{1}{2} + z) = f(\tfrac{1}{2} - z)\,,$$

$$(\mathbf{E}) \qquad f(z+1) = -f(z)\,,$$

were used from the beginning. The question raised by Professor Morley, of Haverford College, whether the properties (**A**, **B**, **C**) were all necessary, led to the formulation of the text which brings the essential elements of that proof into sharper relief.

(B) $$f(2z)\,f(\tfrac{1}{2}) = 2f(z)\,f(z+\tfrac{1}{2})\,.$$

(C) $$f(-z) = -f(z)\,.$$

It will appear that in this system of properties the property (**C**) may be replaced by

(C′) $$\left[\frac{d}{dz}\log\frac{f(z)}{z}\right]_{z=0} = 0\,,$$

a property in itself far less sweeping than (**C**).

That *one* function, the function

$$f(z) = \frac{\sin \pi z}{\pi} = \frac{e^{\pi i z} - e^{-\pi i z}}{2\pi i}\,,$$

possesses the properties (**A**,* **B**, **C**) or (**A**, **B**, **C′**) will be at once granted.

Before showing that *only one* function has the properties, the following lemma will be proved :—

LEMMA. *The most general function $h(z)$ with the properties* (**A**, **B**) is

$$h(z) = e^{az}\,g(z)\,,$$

where $g(z)$ is any particular function with those properties, and where a is an arbitrary constant.

From the general theory† of integral functions it appears at once that we may set

$$h(z) = e^{l(z)}\,g(z)\,, \tag{1_1}$$

where $l(z)$ is any integral function of z, as an expression for the most general function $h(z)$ with the properties ($\mathbf{A}_1$, $\mathbf{A}_2$) ; but $h(z)$ is to satisfy ($\mathbf{A}_3$, **B**) also, whence $l(z)$ must satisfy

$$e^{l(0)} = 1\,, \tag{2_1}$$

$$e^{l(2z)+l(\frac{1}{2})} = e^{l(z)+l(z+\frac{1}{2})}\,. \tag{3_1}$$

As an allowable determination of the constant in $l(z)$ we set

$$l(0) = 0\,. \tag{$2_1'$}$$

From (3_1) we have

$$l(2z) + l(\tfrac{1}{2}) = l(z) + l(z+\tfrac{1}{2}) + \mu_z 2\pi i\,, \tag{$3_1'$}$$

where μ_z is an integer, dependent conceivably upon the particular value of z, but not really so dependent, since the other terms of ($3_1'$) are continuous func-

*(**A**) will always mean ($\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$).

† See the introduction to Part II.

tions of z. Hence we replace μ_z by μ, a constant integer, and determine its value, $\mu = 0$, by substituting in $(3_1')$ $z = 0$; thus,

$$l(2z) + l(\tfrac{1}{2}) = l(z) + l(z + \tfrac{1}{2}). \tag{$3_1''$}$$

$(3_1'')$ is an identity in z. We substitute $z + \frac{1}{2}$ for z and have another identity

$$l(2z + 1) + l(\tfrac{1}{2}) = l(z + \tfrac{1}{2}) + l(z + 1), \tag{$3_1'''$}$$

and by subtraction,

$$l(2z + 1) - l(2z) = l(z + 1) - l(z). \tag{4}$$

Write, for a moment,

$$l(z + 1) - l(z) = k(z). \tag{5}$$

$k(z)$ is an integral function of z with the identity

$$k(2z) = k(z) \tag{4'}$$

or

$$k(z) = k\left[\frac{z}{2}\right] = k\left[\frac{z}{2^2}\right] = \ldots = k\left[\frac{z}{2^n}\right]; \quad (n = 1, 2, 3, \ldots) \tag{4''}$$

whence, since $k(z)$ is a function continuous (in particular) at $z = 0$, $k(z)$ is a constant a,

$$k(z) = a = k(0); \tag{4'''}$$

whence, from (5) and $(2_1')$,

$$l(z + 1) - l(z) = a = l(1); \; l(0) = 0. \tag{6_1}$$

We set

$$l(z) = az + m(z), \tag{7_1}$$

so that $m(z)$ is an integral function of z for which

$$m(0) = 0, \quad m(1) = 0, \tag{2_2}$$

$$m(z + 1) - m(z) = 0, \tag{6_2}$$

$$m(2z) + m(\tfrac{1}{2}) = m(z) + m(z + \tfrac{1}{2}). \tag{3_2}$$

Introducing

$$Z(z) = e^{2\pi i z}, \; 2\pi i z = \log Z, \tag{8}$$

with the initial correspondence

$$Z = 1, \quad z = 0, \tag{9}$$

we set, thus changing the independent variable,

$$m(z) = m_*(Z), \tag{10_1}$$

and have in $m_*(Z)$ an analytic function of Z which in view of (6_2) is single-

valued, and which has singularities, if at all, only at $Z=0$, $Z=\infty$ (which correspond to $z=\infty$), and for which from $(2_2, 3_2)$

$$m_*(1)=0\,, \tag{2_3}$$

$$m_*(Z^2)+m_*(-1)=m_*(Z)+m_*(-Z)\,. \tag{3_3}$$

We may express $m_*(Z)$ as a Laurent series unconditionally convergent for every Z, except perhaps for $Z=0, \infty$,

$$m_*(Z)=\sum_{-\infty}^{+\infty}{}_i\, c_i Z^i,$$

for which the identity

$$\sum_{-\infty}^{+\infty}{}_i\, c_i Z^{2i}+\sum_{-\infty}^{+\infty}{}_i\,(-1)^i c_i=\sum_{-\infty}^{+\infty}{}_i\, c_i Z^i+\sum_{-\infty}^{+\infty}{}_i\,(-1)^i c_i Z^i \tag{3_4}$$

holds. The corresponding coefficients on left and right must be equal. The coefficients of odd powers of Z vanish on left and right. By comparison of the coefficients of Z^{2j} ($j \gtrless 0$) on left and right we have

$$2c_{2j}=c_j\,. \qquad \text{(j is any integer except 0)} \tag{12_1}$$

This recursion formula (12_1) leads to the following determination of the c's with even suffixes $[\gtrless 0]$ in terms of the c's with odd suffixes,

$$c_{2^n\nu}=\frac{1}{2^n}c_\nu\,. \qquad \left(\begin{array}{l}n \text{ is any zero or positive integer,}\\ \nu \text{ is any odd integer.}\end{array}\right) \tag{12_2}$$

At first sight we seem to be almost as far as ever from the definitive determination of $m_*(Z)$, since the identity (3_4) furnishes no information about the coefficients c_ν (ν odd); but fortunately the *convergence* of the Laurent series comes to our assistance, showing that *every coefficient* c_ν (ν *odd*) *vanishes*, and hence (12_2) *every coefficient* c_i ($i \gtrless 0$) *vanishes*, and hence, since $m_*(1)=0$ (2_3), $m_*(Z)$ *vanishes identically*. For, if any particular c_ν (ν odd) were to fail to vanish, consider the partial series obtained by selecting from (11) the infinitude of terms with $i=2^n\nu$, $n=0, 1, 2, \ldots$, which by (12_2) we may write

$$c_\nu \sum_0^\infty{}_n \frac{1}{2^n} Z^{2^n\nu}\,. \tag{13}$$

This power-series should converge for the whole plane, except for $Z=\infty$ (ν positive) or for $Z=0$ (ν negative), and hence the same should be true of its term-by-term derivative series, and of that series multiplied by Z, viz. of the series

$$c_\nu \nu \sum_0^\infty{}_n Z^{2^n\nu}\,; \tag{14}$$

but this series does *not* so converge, it diverges for example for $Z = +1$. Whence the supposition that (any particular) $c_\nu \gtrless 0$ (ν odd) is recognized as untenable.

Thus we have

$$m(z) = m_*(Z) = 0, \tag{10_2}$$

$$l(z) = az, \tag{7_2}$$

$$h(z) = e^{az} g(z). \tag{1_2}$$

The last equality (1_2) completes the proof of the LEMMA; clearly a is an *arbitrary* constant.

If now $g(z)$ is any particular function with the properties (**A**, **B**, **C**), the most general function $h(z)$ with those properties turns out to be $g(z)$ itself, (which is the THEOREM), since the condition (**C**) gives on the a

$$e^{-az} = e^{az}; \tag{15}$$

whence, by a line of argument like that used above with respect to the exponential equation (3_1), we have in fact

$$a = 0, \quad e^{az} = 1. \tag{16}$$

It is obvious that (**C**), which in the system (**A**, **B**, **C**) is needed and used only to effect the determination $a = 0$, may be replaced by (**C′**), so that the systems (**A**, **B**, **C**), (**A**, **B**, **C′**) are equivalent.

PART II.

Application: A new determination of the external exponential factor in the expression of the function $f(z) = \dfrac{\sin \pi z}{\pi}$ *as a Weierstrassian infinite product.*

From a knowledge of the complete system of zeros, multiplicity included, of an integral function $F(z)$ of the complex variable Z, Weierstrass* has shown how to construct in the form of an infinite product of properly determined primary factors an integral function of z with precisely the same system of zeros; the original function $F(z)$ is then the product of Weierstrass's infinite

*Weierstrass: Zur Theorie der eindeutigen analytischen Functionen (Abhandlungen der Königl. Akademie der Wissenchaften zu Berlin vom Jahre 1876). Reprinted in Abhandlungen aus der Functionenlehre, pp. 1–52 (1886).

See also: Forsyth, Theory of Functions of a Complex Variable, chapter V (1893). Harkness and Morley, A Treatise on the Theory of Functions, pp. 186–193 (1893).

product by an integral function which has no zero in the finite z-plane, and which may therefore be expressed in the form

$$e^{l(z)},$$

where $l(z)$ is an integral function of z.

In the application of Weierstrass's method to the factorisation of any particular function this " external exponential factor " demands separate determination. Picard* by use of Cauchy's theorem has effected a determination of the external exponential factor for an extensive system of functions $F(z)$, under which the function $s(z) = \frac{\sin \pi z}{\pi}$ is included. Other determinations for this particular function have been published. The one immediately to be given depends upon the Theorem of Part I, and is, so far as I know, new.

We have the function

$$s(z) = \frac{\sin \pi z}{\pi} = \frac{e^{\pi iz} - e^{-\pi iz}}{2\pi i}, \tag{1}$$

with the properties (**A**, **B**, **C**) of Part I. ($\mathbf{A}_1$, $\mathbf{A}_2$) lead us to the Weierstrassian product

$$p(z) = z\, \Pi'_m \left[1 - \frac{z}{m}\right] e^{\frac{z}{m}}, \quad (m = \pm 1, \pm 2, \ldots) \tag{2}$$

and the equation

$$s(z) = e^{l(z)} p(z)\,. \tag{3}$$

The determination of the factor $e^{l(z)}$, $e^{l(z)} = +1$, will be effected by the direct identification by the Theorem of Part I of $s(z)$ and $p(z)$. $p(z)$ has the properties ($\mathbf{A}_1$, $\mathbf{A}_2$) to begin with, and evidently also ($\mathbf{A}_3$, **C**). It remains then merely to show that $p(z)$ has the property (**B**).

The index m shall take all integral values from $-\infty$ to $+\infty$, $m = 0$ excepted,† the indices n, n' all positive integral values, and the index ν all positive odd integral values. With this understanding, by combining into one factor the two factors of $p(z)$ with $m = +n$, $-n$, we have at once

$$p(z) = z\, \Pi'_m \left[1 - \frac{z}{m}\right] e^{\frac{z}{m}} = z\, \Pi_n \left[1 - \frac{z^2}{n^2}\right]; \tag{4}$$

whence

$$p(2z) = 2z\, \Pi_n \left[1 - \frac{4z^2}{n^2}\right] = 2z\, \Pi_n \left[1 - \frac{z^2}{n^2}\right] \Pi_\nu \left[1 - \frac{4z^2}{\nu^2}\right], \tag{5}$$

where we have distributed the factors of the first infinite product into the two

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* Picard: Traité d'Analyse, vol. II, pp. 164–7, 1893.

† The exception is denoted after Weierstrass by the $'$ in the symbol Π'.

factors according to the evenness or oddness of the n's, $n = 2n'$, $n = 2n' - 1 = \nu$, and then in the first factor replaced the index n' by the equivalent n.

$$p(\tfrac{1}{2}) = \tfrac{1}{2}\, \Pi_n \frac{(2n-1)(2n+1)}{(2n)(2n)}, \tag{6}$$

$$p(z+\tfrac{1}{2}) = \frac{2z+1}{2}\, \Pi_n \left[1 - \frac{(2z+1)^2}{4n^2}\right] \tag{7}$$

$$= \frac{2z+1}{2}\, \Pi_n \frac{(2n-1)(2n+1)}{(2n)(2n)}\, \Pi_n \left[\overline{1 - \frac{2z}{2n-1}} \cdot \overline{1 + \frac{2z}{2n+1}}\right] \tag{7'}$$

$$= \tfrac{1}{2}\, \Pi_n \frac{(2n-1)(2n+1)}{(2n)(2n)}\, \Pi_\nu \left[1 - \frac{4z^2}{\nu^2}\right], \tag{7''}$$

where to pass from (7′) to (7″) we have made use of the identity

$$1 + 2z = \Pi_n \frac{1 + \dfrac{2z}{2n-1}}{1 + \dfrac{2z}{2n+1}}, \tag{8}$$

and have associated the corresponding factors of the second infinite product (7′) and the infinite product (8). From (4, 5, 6, 7″) we have for $p(z)$ the property (**B**)

$$p(2z)\, p(\tfrac{1}{2}) = 2p(z)\, p(z+\tfrac{1}{2}). \tag{9}$$

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